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Boundedness of pseudo-differential operators on the torus

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Introduction

General Context

Harmonic analysis and the theory of pseudo-differential operators constitute fundamental tools in modern harmonic analysis.

- They allow studying the regularity of solutions to partial differential equations.
- They unify the theory of differential operators and Fourier multipliers.
- They provide a framework to quantify boundedness in various function spaces.

In the Euclidean case (\mathbb{R}^n), the theory is well-established thanks to the works of Alberto Calderón, Antoni Zygmund, Lars Hörmander, and Charles Fefferman.

Notation and Terminology

Throughout this presentation we will use the following standard notation:

- **Inequality up to constants:** $A \lesssim B$ indicates that there exists a constant $C > 0$ such that $A \leq CB$. If C depends on a parameter α , we write $A \lesssim_{\alpha} B$.
- **Japanese brackets:** We define $\langle x \rangle := \sqrt{1 + |x|^2}$ to capture the asymptotic behavior without presenting issues at the origin.
- **Multi-indices:** For $\alpha \in \mathbb{N}_0^n$, we denote the partial derivative as $\partial^{\alpha} := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ and its length as $|\alpha| = \sum \alpha_i$.

The Problem in the Euclidean Case

For $0 \leq \delta, \rho \leq 1$, we consider the Hörmander symbol class $S_{\rho, \delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$, consisting of smooth functions $a : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$, which satisfy:

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \lesssim_{\alpha\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}.$$

The associated pseudo-differential operator is defined as:

$$T_a f(x) := \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) \widehat{f}(\xi) d\xi,$$

where \widehat{f} corresponds to the usual Fourier transform in \mathbb{R}^n . Then, we say that $T_a \in \Psi_{\rho, \delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$.

Fefferman's Classical Theorem

One of the central results motivating this thesis is the following:

Theorem (Fefferman [5])

Let $0 \leq \delta < 1 - \varepsilon < 1$, and let $T \in \Psi_{1-\varepsilon,\delta}^m(\mathbb{R}^n \times \mathbb{R}^n)$. Suppose that

$$m \leq -n\varepsilon \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Then the operator T extends to a bounded operator from $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$.

Objectives

Analysis on the torus \mathbb{T}^n presents unique challenges due to the discrete nature of the frequency space (\mathbb{Z}^n). The objectives are:

1. Formalize the symbolic calculus on the torus using discrete finite differences.
2. Extend the L^p boundedness results of Fefferman and Álvarez-Hounie [7] to the toroidal setting.
3. Establish precise conditions for boundedness in the scale of Hardy spaces H^p ($p \leq 1$).
4. Obtain results in weighted spaces A_p according to the techniques of Park and Tomita [10], and Sobolev spaces.

Note that it is not possible to extend the Euclidean results to the torus by means of local charts, because pseudo-differential operators are not stable when $\rho < 1 - \delta$. Furthermore, Hardy and BMO function spaces are not stable under multiplication by test functions, which prevents making a suitable partition of unity.

Preliminaries and Tools

Basic Definitions on the Torus

The n -dimensional torus is defined as the quotient additive group $\mathbb{T}^n := \mathbb{R}^n / \mathbb{Z}^n = (\mathbb{R}/\mathbb{Z})^n$.

The torus can be identified with the set $[0, 1)^n$.

Additionally, it is considered with the quotient topology and the restricted Lebesgue measure.

Lebesgue Spaces $L^p(\mathbb{T}^n)$

Let $w : \mathbb{T}^n \rightarrow \mathbb{C}$ be a non-negative locally integrable function. Then, we can define a measure

$$w(E) := \int_E w(x) dx.$$

Definition (Weighted Lebesgue spaces)

A (strongly) measurable function $f : \mathbb{T}^n \rightarrow X$ is said to belong to the weighted Lebesgue space $L^p(\mathbb{T}^n; X; w)$ when

$$\|f\|_{L^p(w)} := \left(\int_{\mathbb{T}^n} \|f(x)\|_X^p dw(x) \right)^{1/p} < \infty,$$

for $1 \leq p < \infty$. When $p = \infty$, when it is bounded except on a set of w -measure zero.

Weak Type Boundedness

Definition (Weak Type (p, q))

Let T be an operator from $L^p(\Omega; X; w)$ to the space of measurable functions from Σ to Y . T is said to be of weak type (p, q) with respect to the weights (u, w) , with $q < \infty$, if we have:

$$u\{x \in \Sigma : \|Tf(x)\|_Y > \lambda\} \lesssim \left(\frac{\|f\|_{L^p(w)}}{\lambda} \right)^q .$$

This notion is crucial for applying interpolation theorems such as Marcinkiewicz's.

Fourier Analysis on \mathbb{T}^n and \mathbb{Z}^n

Definition (Schwartz Space $\mathcal{S}(\mathbb{Z}^n; X)$)

The space of rapidly decaying functions $\varphi : \mathbb{Z}^n \rightarrow X$ such that for all $M > 0$,

$$\|\varphi(\xi)\|_X \lesssim_M \langle \xi \rangle^{-M}.$$

Definition (Periodic Fourier Transform)

Let $\mathcal{F}_{\mathbb{T}^n} : C^\infty(\mathbb{T}^n; X) \rightarrow \mathcal{S}(\mathbb{Z}^n; X)$ be defined by

$$(\mathcal{F}_{\mathbb{T}^n} f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{T}^n} e^{-i2\pi x \cdot \xi} f(x) dx.$$

Its inverse is

$$(\mathcal{F}_{\mathbb{T}^n}^{-1} \varphi)(x) := \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x \cdot \xi} \varphi(\xi).$$

Distributions on the Torus

Definition (Periodic Distributions $\mathcal{D}'(\mathbb{T}^n)$)

The space of periodic distributions $\mathcal{D}'(\mathbb{T}^n; X)$ consists of the continuous linear operators defined from $C^\infty(\mathbb{T}^n)$ to X .

Every function $f \in L^p(\mathbb{T}^n; X)$ is a periodic distribution via the functional defined as

$$\langle f, \varphi \rangle := \int_{\mathbb{T}^n} f(x) \varphi(x) dx.$$

Theorem (Discrete Distributions $\mathcal{S}'(\mathbb{Z}^n; X)$)

The continuous linear operators defined from $\mathcal{S}(\mathbb{Z}^n)$ to X have the form

$$\langle u, \varphi \rangle = \sum_{\xi \in \mathbb{Z}^n} u(\xi) \varphi(\xi).$$

Distributional Analysis

Definition (Distributional Derivative)

For $u \in \mathcal{D}'(\mathbb{T}^n)$, we define its distributional derivative $\partial^\alpha u$ by:

$$\langle \partial^\alpha u, \varphi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \varphi \rangle \quad .$$

Definition (Distributional Fourier Transform)

Furthermore, the distributional Fourier transform $\mathcal{F} : \mathcal{D}'(\mathbb{T}^n; X) \rightarrow \mathcal{S}'(\mathbb{Z}^n; X)$ is defined as

$$\langle \mathcal{F}u, \varphi \rangle := \langle u, \iota \circ \mathcal{F}^{-1}\varphi \rangle.$$

Where $(\iota \circ f)(x) = f(-x)$.

Sobolev Spaces

Definition (Sobolev Spaces)

Let $1 \leq p \leq \infty$ and let $k \in \mathbb{N}_0$. The Sobolev space $W_p^k(\mathbb{T}^n; X)$ consists of all functions $f \in L^p(\mathbb{T}^n; X)$ such that for any multi-index $|\alpha| \leq k$, $\partial^\alpha f$ exists (in the sense of distributions) and belongs to $L^p(\mathbb{T}^n; X)$. For such functions we define

$$\|f\|_{W_p^k} := \left(\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p}^p \right)^{1/p}, \quad (1 \leq p < \infty).$$

Hardy and BMO Spaces

Hardy Space H^1

The Fefferman-Stein theory [12] is adapted. With $\Omega = \mathbb{R}^n, \mathbb{T}^n$.

Definition (Hardy Space H^1)

It is said that $f \in L^1(\Omega; X)$ is in the Hardy space $H^1(\Omega; X)$ if there exist $f_1, \dots, f_n \in L^1(\Omega; X)$ satisfying

$$\widehat{f}_j(\xi) = \frac{i\xi_j}{|\xi|} \widehat{f}(\xi).$$

We write $f_j =: R_j f$ (Riesz transform), and define the norm

$$\|f\|_{H^1} := \|f\|_{L^1} + \sum_{j=1}^n \|R_j f\|_{L^1}.$$

Atomic Hardy Spaces H_{at}^p

For $0 < p \leq 1$, we define $H^p(\mathbb{T}^n)$ by means of (p, q) -atoms.

Definition $((p, q)$ -Atom)

Let $1 \leq q \leq \infty$ and $p \leq 1$. A function $a(x)$ is a (p, q) -atom if there exists a ball B such that:

1. $\text{supp } a \subset B$.
2. $\|a\|_{L^q} \leq |B|^{1/q-1/p}$.
3. *Vanishing moments:* $\int x^\beta a(x) dx = 0$ for $0 \leq |\beta| \leq n(1/p - 1)$.

Equivalence of Atomic Definitions

Theorem

We have that $H_{at}^{p,q}(\Omega; X) = H_{at}^{p,r}(\Omega; X)$ for any $1 \leq q, r \leq \infty$, with equivalent norms.

In particular, $H^1(\Omega; X) = H_{at}^{1,\infty}(\Omega; X)$. This justifies using (p, ∞) or $(p, 2)$ atoms interchangeably to define H^p .

BMO Space

Definition (BMO Space)

f is said to belong to the space of bounded mean oscillation functions $\text{BMO}(\Omega; X)$ if the sharp maximal operator $f^\# \in L^\infty$, where:

$$f^\#(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q \|f(y) - f_Q\|_X dy.$$

We define the norm $\|f\|_{\text{BMO}} := \|f^\#\|_{L^\infty}$.

H^1 -BMO Duality

This result is fundamental for complex interpolation.

Theorem (Fefferman [6])

Suppose X' satisfies the Radon-Nikodym property. Then, the dual of $H^1(\Omega; X)$ is $BMO(\Omega; X')$, with $\Omega = \mathbb{R}^n, \mathbb{T}^n$.

- 1. For $\varphi \in BMO$, the functional $f \mapsto \int_{\Omega} \varphi(x)f(x) dx$ is bounded on H^1 .*
- 2. Any continuous functional on H^1 behaves like the functional in (1) for a unique function $\varphi \in BMO$.*

Complex Interpolation

Now, we present the complex interpolation argument of Fefferman-Stein [12].

Theorem

Let $z \mapsto T_z$ be an analytic family of operators. Fix $1/p = 1 - \theta/2$ with $0 < \theta < 1$.

1. If $\sup_y \|T_{iy}f\|_{L^1} \lesssim \|f\|_{H^1}$, and $\sup_y \|T_{1+iy}f\|_{L^2} \lesssim \|f\|_{L^2}$, then

$$\|T_\theta f\|_{L^p} \lesssim \|f\|_{L^p}.$$

2. If $\sup_y \|T_{iy}f\|_{\text{BMO}} \lesssim \|f\|_{L^\infty}$, and $\sup_y \|T_{1+iy}f\|_{L^2} \lesssim \|f\|_{L^2}$, then

$$\|T_\theta f\|_{L^{p'}} \lesssim \|f\|_{L^{p'}}.$$

Pseudo-differential Operators on the Torus

Discrete Finite Differences

We follow the framework developed by Ruzhansky and Turunen [11]. To define symbol classes on the lattice \mathbb{Z}^n , we replace derivatives with finite differences.

Definition

Let $\varphi : \mathbb{Z}^n \rightarrow \mathbb{C}$, then the difference operators are defined as

$$\Delta_{\xi_j} \varphi(\xi) := \varphi(\xi + \delta_j) - \varphi(\xi),$$

$$\bar{\Delta}_{\xi_j} \varphi(\xi) := \varphi(\xi) - \varphi(\xi - \delta_j).$$

Moreover, for a multi-index $\alpha \in \mathbb{N}_0^n$, we define $\Delta_{\xi}^{\alpha} := \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}$.

Toroidal Symbol Class

Definition (Toroidal symbol class $S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$)

Let $m \in \mathbb{R}$, let $0 \leq \delta, \rho \leq 1$. Then, the toroidal symbol class $S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ consists of functions $a := a(x, \xi) : \mathbb{T}^n \times \mathbb{Z}^n \rightarrow \mathbb{C}$ that are smooth in x for all ξ , and satisfy the symbolic inequalities

$$|\Delta_{\xi}^{\alpha} \partial_x^{\beta} a(x, \xi)| \lesssim_{\alpha\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|},$$

for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$.

Toroidal Pseudo-differential Operators

Definition (Toroidal pseudo-differential operators)

For $a \in S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, T_a denotes its corresponding toroidal pseudo-differential operator, defined as

$$T_a f(x) := \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi x \cdot \xi} a(x, \xi) \widehat{f}(\xi).$$

Furthermore, we say that $T_a \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$.

For $f \in C^\infty(\mathbb{T}^n)$, the series converges absolutely and $T_a f \in C^\infty(\mathbb{T}^n)$.

The Toroidal Schwartz Kernel

The definition of T_a suggests the integral representation:

$$T_a f(x) = \int_{\mathbb{T}^n} k(x, y) f(y) dy,$$

where $k(x, y)$ is the *Schwartz kernel* expressed in the sense of distributions as:

$$k(x, y) := \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi(x-y)\cdot\xi} a(x, \xi).$$

Symbol Equivalence Theorem

This result is crucial as it connects the discrete theory with the continuous one.

Theorem (Ruzhansky, Turunen [11])

Let $0 \leq \delta \leq 1$ and $0 < \rho \leq 1$. The symbol $\tilde{a} \in S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ is a toroidal symbol if and only if there exists a Euclidean symbol $a \in S_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{R}^n)$ such that $\tilde{a} = a|_{\mathbb{T}^n \times \mathbb{Z}^n}$. Furthermore, this extension is unique modulo $S^{-\infty}(\mathbb{T}^n \times \mathbb{R}^n)$.

Moreover, the corresponding pseudo-differential operators coincide.

Boundedness in Lebesgue spaces

Kernel Properties for L^p

Theorem (Cardona, M., JMAA [1])

Let $T \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ with kernel $k(x, y)$.

1. k is smooth off the diagonal.
2. Given $\alpha, \beta \in \mathbb{N}_0^n$, for $N > (m + n + |\alpha + \beta|)/\rho$ we have:

$$\sup_{x \neq y} |x - y|^N |\partial_x^\alpha \partial_y^\beta k(x, y)| = C_{\alpha\beta N} < \infty.$$

This polynomial decay is essential to prove boundedness in L^p spaces using classical theory. The proof is based on integration by parts and the decay properties of the symbol.

$L^2(\mathbb{T}^n)$ Boundedness

The following result is analogous to that of Jorge Hounie [9]. Here and in the sequel, $\lambda := \max\{(\delta - \rho)/2, 0\}$.

Theorem (Cardona, M., JMAA [1])

Let $\tilde{p} : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ be a symbol such that for $0 < \rho \leq 1$, $0 \leq \delta < 1$, $m \leq -n\lambda$ and $|\alpha|, |\beta| \leq \lceil n/2 \rceil$, it satisfies:

$$\left| \partial_\xi^\alpha \partial_x^\beta \tilde{p}(x, \xi) \right| \leq C_{\alpha\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}.$$

Then $T_{\tilde{p}}$ is bounded from $L^2(\mathbb{T}^n)$ to itself.

Kernel Estimates

To prove boundedness in L^p , kernel estimates are obtained to use an argument similar to the classical one by Fefferman and Stein.

Theorem (Cardona, M., JMAA [1])

Let $T \in \Psi_{\rho,\delta}^m$ with the usual conditions.

- If $\sigma \geq \varepsilon$,

$$\sup_{|y-z| \leq \sigma} \int_{|x-z| > 2\sigma} |k(x, y) - k(x, z)| dx \leq C_\varepsilon.$$

- If $m \leq -n[(1 - \rho)/2 + \lambda]$ and $\sigma < 1$:

$$\sup_{|y-z| \leq \sigma} \int_{|x-z| > 2\sigma^\rho} |k(x, y) - k(x, z)| dx \leq C.$$

Estimates in the other kernel variable are also valid.

Weak Type (1,1) Boundedness

Theorem (Cardona, M., JMAA[1])

Let $T \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, with $0 < \rho \leq 1$, $0 \leq \delta < 1$, $m \leq -n[(1 - \rho)/2 + \lambda]$, then T is of weak type $(1, 1)$.

This is proved using a Calderón-Zygmund decomposition and an argument with convolutions similar to Fefferman's. This result was proved for the general case of operators with operator-valued kernels analogous to Álvarez and Milman [8].

Boundedness $H^1 \rightarrow L^1$ and $L^\infty \rightarrow \text{BMO}$

Theorem (Cardona, M., JMAA [1])

Let $T \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, with $0 < \rho \leq 1$, $0 \leq \delta < 1$. If $m \leq -n[(1 - \rho)/2 + \lambda]$, then T and its adjoint T^* are bounded maps:

- from the Hardy space $H^1(\mathbb{T}^n)$ to $L^1(\mathbb{T}^n)$,
- from $L^\infty(\mathbb{T}^n)$ to $\text{BMO}(\mathbb{T}^n)$.

Note that $\delta < \rho$ is no longer required as in Fefferman's result.

Sketch of the Proof: $H^1 \rightarrow L^1$

Let a be an atom in $B(z, \sigma)$. We want to uniformly bound $\|Ta\|_{L^1}$. We assume $\sigma < 1$, the complementary case is analogous.

1. **Decomposition:** We split the integral into $B'(z, 2\sigma^\rho)$ (local part) and its complement (far part).
2. **Local Part:** We use Hölder and $L^q \rightarrow L^2$ boundedness.

$$\int_{B'} |Ta| \lesssim |B'|^{1/2} \|Ta\|_{L^2} \lesssim |B'|^{1/2} \|a\|_{L^{2/(2-\rho)}} \leq C.$$

3. **Far Part:** We use the cancellation property of the atom ($\int a = 0$) and the kernel estimates from the previous theorem to bound:

$$\int_{\mathbb{T}^n \setminus B'} |Ta(x)| dx \leq \int_{\mathbb{T}^n \setminus B'} \int_B |k(x, y) - k(x, z)| |a(y)| dy dx \leq C.$$

Sketch of the Proof: $L^\infty \rightarrow \text{BMO}$

Let $f \in L^\infty$. For each ball $B(z, \sigma)$, we decompose $f = f\chi_{B'} + f\chi_{(B')^c} = f_1 + f_2$. Again we assume $\sigma < 1$.

1. **f_1 Term (Local):** We use that T is bounded on $L^2 \rightarrow L^{2/\rho}$.

$$\frac{1}{|B|} \int_B |Tf_1| \lesssim |B|^{-\rho/2} \|Tf_1\|_{L^{2/\rho}} \lesssim |B|^{-\rho/2} \|f_1\|_{L^2} \lesssim \|f\|_{L^\infty}.$$

2. **f_2 Term (Far):** For $x \in B$, $Tf_2(x)$ is a smooth function. We choose the constant $c_B = Tf_2(z)$ (center of the ball).

$$\frac{1}{|B|} \int_B |Tf_2(x) - c_B| dx \leq \sup_{x \in B} \int_{(B')^c} |k(x, y) - k(z, y)| |f(y)| dy \lesssim \|f\|_{L^\infty}.$$

Main Theorem: $L^p(\mathbb{T}^n)$ Boundedness

Using the complex interpolation argument between the endpoint results (H^1, L^1) - (L^2, L^2) and (L^2, L^2) - (L^∞, BMO) :

Theorem (Cardona, M., JMAA [1])

Let $T \in \Psi_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, with $0 < \rho \leq 1$, $0 \leq \delta < 1$ and

$$m \leq -n \left[(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right| + \lambda \right].$$

Then T is a bounded map from $L^p(\mathbb{T}^n)$ to itself.

Boundedness $L^p \rightarrow L^q$

Using Bessel potentials and the Hardy-Littlewood-Sobolev inequality, it can be extended to

Theorem (Cardona, M., JMAA [1])

Let $T \in \Psi_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$. T is bounded from $L^p(\mathbb{T}^n)$ to $L^q(\mathbb{T}^n)$ if:

- Case $1 < p \leq 2 \leq q$: $m \leq -n(1/p - 1/q + \lambda)$.
- Case $2 \leq p \leq q$: $m \leq -n[1/p - 1/q + (1 - \rho)(1/2 - 1/p) + \lambda]$.
- Case $p \leq q \leq 2$: $m \leq -n[1/p - 1/q + (1 - \rho)(1/q - 1/2) + \lambda]$.

Boundedness in Sobolev spaces

Sobolev spaces revisited

The Bessel potential J^s is defined as the pseudo-differential operator with symbol $\langle \xi \rangle^s$.

Definition

It is said that $f \in W_p^s(\mathbb{T}^n)$, if we have $J^s f \in L^p(\mathbb{T}^n)$. The norm is defined as $\|f\|_{W_p^s} := \|J^s f\|_{L^p}$.

This definition coincides with the one presented earlier when s is a positive integer.

Boundedness in Sobolev Spaces

Theorem (Cardona, M. [3])

Let $0 \leq \delta < 1$, $0 < \rho \leq 1$, $m \in \mathbb{R}$, and $T \in \Psi_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$. Then, T extends to a bounded operator from $W_p^s(\mathbb{T}^n)$ to $W_q^{s-\mu}(\mathbb{T}^n)$ for $1 < p \leq q < \infty$ if:

- Case $1 < p \leq 2 \leq q$: $\mu \geq m + n(1/p - 1/q + \lambda)$.
- Case $2 \leq p \leq q$: $\mu \geq m + n[1/p - 1/q + (1 - \rho)(1/2 - 1/p) + \lambda]$.
- Case $p \leq q \leq 2$: $\mu \geq m + n[1/p - 1/q + (1 - \rho)(1/q - 1/2) + \lambda]$.

Proof

The proof uses the composition property with Bessel potentials J^s .

$$\|Tf\|_{W_q^{s-\mu}} = \|J^{s-\mu}Tf\|_{L^q} = \|(J^{s-\mu}TJ^{-s})J^s f\|_{L^q}.$$

The composed operator $S = J^{s-\mu}TJ^{-s}$ is a pseudo-differential operator of order $m - \mu$. If the conditions of the $L^p \rightarrow L^q$ theorem hold for order $m - \mu$, then S is bounded from L^p to L^q .

$$\|S(J^s f)\|_{L^q} \lesssim \|J^s f\|_{L^p} = \|f\|_{W_p^s}.$$

Weighted Results

Maximal operators

Definition

The Hardy-Littlewood p -maximal operator is defined as

$$M_p f(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(x)|^p dx \right)^{1/p}.$$

Definition

And the Fefferman-Stein sharp p -maximal operator as

$$\mathcal{M}_p^\# f(x) := \sup_{Q \ni x} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p},$$

where f_Q is the average over the cube Q .

Muckenhoupt A_p Weight Classes

Definition (Muckenhoupt weight classes)

For a pair of locally integrable non-negative functions $u, w : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{C}$, they are said to belong to the Muckenhoupt weight class A_p , if

$$Mu(x) \lesssim w(x), \quad \text{for almost every } x, \quad p = 1;$$

$$\sup_Q \left(\frac{1}{|Q|} \int_Q u(x) \, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} \, dx \right)^{p-1} < \infty, \quad 1 < p < \infty.$$

The Hardy-Littlewood maximal operator M is of weak type (p, p) with respect to w if and only if $w \in A_p$.

Pointwise Inequality

We proceed as in Park and Tomita [10].

Theorem (Cardona, M. [3])

Let $1 < r \leq 2$, $0 < \rho < 1$ and suppose that $\sigma \in S_{\rho, \rho}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ with $m \leq -n(1 - \rho)/r$. Then, for all $f \in C^\infty(\mathbb{T}^n)$:

$$\mathcal{M}_r^\#(T_\sigma f)(x) \lesssim \mathbf{M}_r f(x).$$

This implies boundedness on $L^p(w)$ for $w \in A_{p/r}$:

$$\|T_\sigma f\|_{L^p(w)} \lesssim \|\mathcal{M}_r^\# f\|_{L^p(w)} \lesssim \|\mathbf{M}_r f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$

Boundedness in Hardy Spaces H^p

Kernel Estimates

In this case, estimates on dyadic annular decompositions are obtained:

$$A_j(z, \sigma) := \{x \in \mathbb{T}^n : 2^j \sigma < |x - z| \leq 2^{j+1} \sigma\}.$$

Theorem (Cardona, M. [2])

Let $T \in \Psi_{\rho, \delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$ with the usual conditions.

- If $\sigma \geq \varepsilon$,

$$\sup_{|y-z| \leq \sigma} \int_{A_j(z, \sigma)} |k(x, y) - k(x, z)| \, dx \leq C_\varepsilon 2^{-j}.$$

- If $m \leq -n[(1 - \rho)/2 + \lambda]$ and $\sigma < 1$:

$$\sup_{|y-z| \leq \sigma} \int_{A_j(z, \sigma^\gamma)} |k(x, y) - k(x, z)| \, dx \leq C 2^{-j/\rho} \sigma^{1-\gamma/\rho}.$$

These estimates are also valid in the other variable.

Boundedness $H^p \rightarrow L^p$

Theorem (Cardona, M. [2])

Let $T \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 < \rho \leq 1$, $0 \leq \delta < 1$. Suppose that

$$m \leq -\beta - n\lambda \quad \text{for some} \quad (1 - \rho)\frac{n}{2} \leq \beta < \frac{n}{2}.$$

Then, the operator T is bounded from $H^p(\mathbb{T}^n)$ to $L^p(\mathbb{T}^n)$ for $1 \geq p \geq p_0$ where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta(1/\rho + n/2)}{n(1/\rho - 1 + \beta)}.$$

Concept of a Molecule

The image of an atom Ta does not have compact support. We introduce the concept of a molecule.

Definition $((p, \theta, \mu)$ -Molecule)

A function $M(x)$ associated to a ball $B(z, \sigma)$ is a molecule if it satisfies $\int M = 0$ and decay conditions. If $\sigma \geq 1$:

- $\int \|M(x)\|_Y^2 dx \lesssim \sigma^{n(1-2/p)}$.
- $\int \|M(x)\|_Y^2 |x - z|^\mu dx \lesssim \sigma^{\mu+n(1-2/p)}$.

And if $\sigma < 1$:

- $\int \|M(x)\|_Y^2 dx \lesssim \sigma^{n(1/q-2/p)}$.
- $\int \|M(x)\|_Y^2 |x - z|^\mu dx \lesssim \sigma^{\theta\mu+n(1/q-2/p)}$.

Here θ and q are parameters.

Boundedness $H^p \rightarrow H^p$

Lemma

If M is a molecule with μ sufficiently large, then $M \in H^p$ and its norm only depends on the constants from the molecule conditions.

Theorem (Cardona, M. [2])

Let $T \in \Psi_{\rho,\delta}^m(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 < \rho \leq 1$, $0 \leq \delta < 1$. Suppose that

$$m \leq -\beta - n\lambda \quad \text{for some} \quad (1 - \rho)\frac{n}{2} \leq \beta < \frac{n}{2}.$$

If additionally $T^*(1) = 0$ (in the sense of BMO), then T is bounded from $H^p(\mathbb{T}^n)$ to itself for $p_0 < p \leq 1$, where

$$\frac{1}{p_0} = \frac{1}{2} + \frac{\beta(1/\rho + n/2)}{n(1/\rho - 1 + \beta)}.$$

Sketch of Proof: $H^p \rightarrow H^p$

This theorem was also proved in the context of operators with operator-valued kernels.

1. **Atoms:** Let a be a $(p, 2)$ -atom. We want to show that Ta has a uniform H^p norm.
2. **Molecules:** It is shown that under the hypotheses of the theorem, $M = Ta$ satisfies the conditions of a (p, θ, μ) -molecule, with constants depending only on the operator.
 - The L^2 and L^q estimates of T control the L^2 norm of M near the support of the atom.
 - The kernel estimates control the decay of M away from the support.
3. **Cancellation:** We use the condition $T^*(1) = 0$:

$$\int Ta = \langle 1, Ta \rangle = \langle T^*(1), a \rangle = 0.$$

4. **Conclusion:** Since Ta is a molecule, $\|Ta\|_{H^p} \leq C$.

Selected References (I)



Cardona, D. and Martínez, M. A.: *Estimates for pseudo-differential operators on the torus revisited. I.* J. Math. Anal. Appl., 2026.



Cardona, D. and Martínez, M. A.: *Estimates for pseudo-differential operators on the torus revisited. II.* arXiv:2505.01573, 2025.



Cardona, D. and Martínez, M. A.: *Estimates for pseudo-differential operators on the torus revisited. III.* arXiv:2508.13338, 2025.

Selected References (II)

-  Delgado, J.: *L^p bounds for pseudo-differential operators on the torus*. Operator Theory: Advances and Applications, 2012.
-  Fefferman, C.: *L^p bounds for pseudo-differential operators*. Israel Journal of Mathematics, 1973.
-  Fefferman, C.: *Characterizations of bounded mean oscillation*. Bulletin of the American Mathematical Society, 1971.
-  Álvarez, J. and Hounie, J.: *Estimates for the kernel and continuity properties of pseudo-differential operators*. Arkiv för Matematik, 1990.
-  Álvarez, J. and Milman, M.: *Vector valued inequalities for strongly singular Calderón-Zygmund operators*. Revista Matemática Iberoamericana, 2(4):405–426, 1986.

Selected References (III)

-  Hounie, J.: *On the L^2 continuity of pseudo-differential operators*. Communications in Partial Differential Equations, 11(7):765–778, 1986.
-  Park, B. J. and Tomita, N.: *Sharp maximal function estimates for linear and multilinear pseudo-differential operators*. Journal of Functional Analysis, 287(12), 2024
-  Ruzhansky, M. and Turunen, V.: *Pseudo-differential operators and symmetries: Background analysis and advanced topics*. Birkhäuser, Basel, 2010
-  Fefferman, C. and Stein, E. M.: *H^p spaces of several variables*. Acta Mathematica, 129(0):137–193, 1972.
-  Stein, E. M.: *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton Univ. Press, 1993.

Thank you for your attention!